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de Snoo, Hendrik; Winkler, Henrik; Wojtylak, Michal

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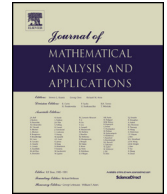
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Global and local behavior of zeros of nonpositive type

Henk de Snoo^a, Henrik Winkler^b, Michał Wojtylak^{c,*,1}^a *Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, Netherlands*^b *Institut für Mathematik, Technische Universität Ilmenau, Curiebau, Weimarer Str. 25, 98693 Ilmenau, Germany*^c *Institute of Mathematics, Jagiellonian University, ul. Łojasiewicza 6, 30-348 Kraków, Poland*

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ABSTRACT

A generalized Nevanlinna function $Q(z)$ with one negative square has precisely one generalized zero of nonpositive type in the closed extended upper halfplane. The fractional linear transformation defined by $Q_\tau(z) = (Q(z) - \tau)/(1 + \tau Q(z))$, $\tau \in \mathbb{R} \cup \{\infty\}$, is a generalized Nevanlinna function with one negative square. Its generalized zero of nonpositive type $\alpha(\tau)$ as a function of τ is being studied. In particular, it is shown that it is continuous and its behavior in the points where the function extends through the real line is investigated.

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1. Introduction

Let $M(z)$ be an ordinary Nevanlinna function, i.e., a function which is holomorphic in \mathbb{C}^+ and which maps the upper half-plane into itself. It is well known that $M(z)$ admits a representation

$$M(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (1)$$

with $a \in \mathbb{R}$, $b > 0$, and a measure σ satisfying $\int_{\mathbb{R}} d\sigma(t)/(t^2+1) < \infty$; cf. [7]. In the lower half-plane \mathbb{C}^- the function $M(z)$ is defined by the symmetry principle $\overline{M(\bar{z})} = M(z)$. Then $M(z)$ is holomorphic on $\mathbb{C}^+ \cup \mathbb{C}^- \cup (\mathbb{R} \setminus \text{supp } \sigma)$. Note that if $\mathbb{R} \setminus \text{supp } \sigma$ contains some interval I , then the extension of $M(z)$ given on \mathbb{C}^+ to the set $\mathbb{C}^+ \cup \mathbb{C}^- \cup I$ is given by the Schwarz reflection principle.

However, the main interest in this paper is in the situation when $M(z)$, restricted to \mathbb{C}^+ , has a holomorphic continuation $\widetilde{M}(z)$ across an interval $I \subset \mathbb{R}$ without $\widetilde{M}(z)$ being real for $z \in I$. In this case the symmetry

* Corresponding author.

E-mail addresses: desnoo@math.rug.nl (H. de Snoo), henrik.winkler@tu-ilmenau.de (H. Winkler), michal.wojtylak@gmail.com (M. Wojtylak).¹ The work of Michał Wojtylak was supported by the Alexander von Humboldt Foundation and performed during his research stay at TU Berlin.

property will be lost. For example the constant function

$$M(z) = i = \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) dt, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

as defined on \mathbb{C}^+ , extends to a holomorphic function on all of \mathbb{C} , although $\text{supp } \sigma = \mathbb{R}$.

Let $Q(z)$ be a generalized Nevanlinna function of class \mathbf{N}_1 , i.e., a meromorphic function in the upper half-plane such that the kernel

$$\mathbf{N}_Q(z, w) = \frac{Q(z) - \overline{Q(w)}}{z - \bar{w}}, \quad z, w \in \mathbb{C}^+,$$

has precisely one negative square. It has been shown that $Q(z)$ has a unique factorization

$$Q(z) = R(z)M(z), \tag{2}$$

with $R(z)$ of one of the following three forms

$$\frac{(z-\alpha)(z-\bar{\alpha})}{(z-\beta)(z-\bar{\beta})}, \quad (z-\alpha)(z-\bar{\alpha}), \quad \frac{1}{(z-\beta)(z-\bar{\beta})}, \tag{3}$$

with $M(z)$ being a Nevanlinna function and $\alpha, \beta \in \mathbb{C}^+ \cup \mathbb{R}$; cf. [4,6]. The point α is called the *generalized zero of nonpositive type* (GZNT) of $Q(z)$ and the point β is called the *generalized pole of nonpositive type* (GPNT) of $Q(z)$; see e.g. [4,6,10] for a characterization of GZNT and GPNT in terms of nontangential limits. The extensions of \mathbf{N}_1 functions that arise from not necessarily symmetric extensions of $M(z)$ are the main objects of the paper.

A function $Q(z)$ in \mathbf{N}_1 generates of family of functions $Q_\tau(z)$ via the linear fractional transformation

$$Q_\tau(z) := \frac{Q(z) - \tau}{1 + \tau Q(z)}, \quad \tau \in \mathbb{R},$$

and by

$$Q_\infty(z) := -\frac{1}{Q(z)}, \quad \tau = \infty.$$

It is known that $Q_\tau(z) \in \mathbf{N}_1$, which allows to define for $\tau \in \mathbb{R} \cup \{\infty\}$ the numbers $\alpha(\tau)$ and $\beta(\tau)$ as, respectively, GZNT and GPNT of the function $Q_\tau(z)$. The local properties of $\alpha(\tau)$ in the case when $\alpha(\tau_0)$ lies in a spectral gap of $M(z)$ were investigated in detail in [15]. In the present work these results are generalized to the case when $Q(z)$ extends holomorphically to the lower half-plane around $\alpha(\tau_0)$. The paper also contains some results of a global nature concerning the function $\tau \rightarrow \alpha(\tau)$. In particular, in [Theorem 3.2](#) it is shown that $\alpha(\tau)$ forms a curve on the Riemann sphere which is homeomorphic to a circle. This problem was still open in [15] and is now solved by means of recent results concerning the convergence behavior of generalized Nevanlinna functions [11]. The problem is related to the convergence of poles in Padé approximation, see [3,14]. A related problem of tracking the eigenvalue of nonpositive type in the context of random matrices was considered in [13,16].

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2. Non-symmetric extensions of Nevanlinna functions

The following result, playing a crucial role in the paper, can be found in [8].

Theorem 2.1. *Let $M(z)$ be a Nevanlinna function of the form (1) and let Ω be a simply connected domain, symmetric with respect to \mathbb{R} . Then the following statements are equivalent:*

- (i) *the restriction of $M(z)$ to \mathbb{C}^+ extends to a holomorphic function in $\Omega \cup \mathbb{C}^+$;*
- (ii) *the measure $d\sigma$ in (1) satisfies*

$$d\sigma(t) = \phi(t) dt, \quad t \in \Omega \cap \mathbb{R},$$

where $\phi(z)$ is a real holomorphic function on Ω .

The above result will be now extended to \mathbf{N}_1 -functions.

Theorem 2.2. *Let $Q(z)$ be in \mathbf{N}_1 with the representation $Q(z) = R(z)M(z)$ as in (2) and (3) with $\alpha, \beta \in \mathbb{C}^+ \cup \mathbb{R}$. Let Ω be a simply connected domain, symmetric with respect to \mathbb{R} and assume that $\beta \notin \Omega$ and $\alpha \in \Omega \cap \mathbb{R}$. Then the following statements are equivalent:*

- (i) *the restriction of $Q(z)$ to $\mathbb{C}^+ \setminus \{\beta\}$ extends to a holomorphic function in $\Omega \cup \mathbb{C}^+ \setminus \{\beta\}$;*
- (ii) *the function $M(z)$ is of the form*

$$M(z) = M_1(z) + \frac{m_0}{\alpha - z}, \quad t \in \Omega \cap \mathbb{R}, \quad (4)$$

where $M_1(z)$ is a Nevanlinna function such that the restriction of $M_1(z)$ to \mathbb{C}^+ extends to a holomorphic function in $\mathbb{C}^+ \cup \Omega$ and $m_0 \geq 0$;

- (iii) *the measure $d\sigma$ for $M(z)$ in (1) satisfies*

$$d\sigma(t) = \phi(t) dt + m_0 \delta_\alpha(t), \quad t \in \Omega \cap \mathbb{R}, \quad (5)$$

where $\phi(z)$ is a real holomorphic function on Ω , $m_0 \geq 0$, and δ_α is the Dirac measure at α .

If instead of $\alpha \in \Omega \cap \mathbb{R}$ one assumes $\alpha \notin \Omega$, then the equivalences above hold with $m_0 = 0$ in statements (ii) and (iii).

Proof. The proof will be given for the case $\alpha \in \Omega \cap \mathbb{R}$; the proof in the case $\alpha \notin \Omega$ is left as an exercise.

(i) \Rightarrow (ii) Let $\tilde{Q}(z)$ be a holomorphic extension of $Q(z)$ to $(\mathbb{C}^+ \cup \Omega) \setminus \{\beta\}$ and let

$$\tilde{M}(z) = R_0^{-1}(z) \frac{1}{(z - \alpha)^2} \tilde{Q}(z),$$

where $R_0(z) = 1/(z - \beta)(z - \bar{\beta})$ or $R_0(z) \equiv 1$, depending on the position of the GPNT β . Note that $\tilde{M}(z)$ is holomorphic in $(\Omega \cup \mathbb{C}^+) \setminus \{\alpha\}$, α is a pole of $\tilde{M}(z)$ of order at most two of $\tilde{M}(z)$ and $\tilde{M}(z) = M(z)$ for $z \in \mathbb{C}^+$. Since $M(z)$ is a Nevanlinna function the limit

$$m_0 = \lim_{z \rightarrow \alpha} (\alpha - z)M(z)$$

exists and is nonnegative. Consequently, α is a pole of $\tilde{M}(z)$ of order at most one, with the residuum m_0 . Therefore, the function

$$M_1(z) = M(z) - \frac{m_0}{\alpha - z} \quad \text{with } m_0 = \lim_{z \rightarrow \alpha} (\alpha - z)M(z), \quad (6)$$

is a Nevanlinna function, see [7, Chapter II.2]. Put

$$\widetilde{M}_1(z) = \widetilde{M}(z) - \frac{m_0}{\alpha - z},$$

and note that

$$\lim_{z \rightarrow \alpha} (\alpha - z) \widetilde{M}_1(z) = 0.$$

Thus, $\widetilde{M}_1(z)$ is holomorphic at α and in consequence in $\mathbb{C}^+ \cup \Omega$.

The implication (ii) \Rightarrow (i) is obvious and the equivalence (ii) \Leftrightarrow (iii) is a direct consequence of [Theorem 2.1](#). \square

3. Global properties of the function $\alpha(\tau)$

Before continuing with extension properties across \mathbb{R} an open problem from [15] will be solved. For this aim consider the following definition. Let \mathcal{D} be a nonempty open subset of the complex plane, and let (Q_n) be a sequence of functions which are meromorphic on \mathcal{D} . The sequence (Q_n) is said to *converge locally uniformly on \mathcal{D}* to the function Q , if for each nonempty open set $\mathcal{D}_0 \subseteq \mathbb{C}$ with compact closure $\overline{\mathcal{D}}_0 \subseteq \mathcal{D}$ there exists an index $n_0(\mathcal{D}_0)$ such that for $n > n_0(\mathcal{D}_0)$ the functions (Q_n) are holomorphic on \mathcal{D}_0 and

$$\lim_{n \rightarrow \infty} Q_n(z) = Q(z), \quad \text{uniformly on } \mathcal{D}_0.$$

(In other words, (Q_n) converges to Q in the compact-open topology on the space of holomorphic functions on \mathcal{D} with values in the extended complex plane $\overline{\mathbb{C}}$, see [1,2].) The reader is referred to [11] for a treatment on locally uniform convergence of \mathbf{N}_κ -functions. The proof of the following result can be derived from [1, Chapter 2.8]; an elementary argument is included anyway.

Proposition 3.1. *Let $Q(z)$ be an \mathbf{N}_1 function and let $\tau_n \in \mathbb{R}$ converge to $\tau \in \overline{\mathbb{R}}$. Then $Q_{\tau_n}(z)$ converges locally uniformly to $Q_\tau(z)$ on $\mathbb{C}^+ \setminus \{\beta(\tau)\}$.*

Proof. Let \mathcal{D} be some open, bounded subset of \mathbb{C}^+ with $\overline{\mathcal{D}} \subset \mathbb{C}^+ \setminus \{\beta(0), \beta(\tau)\}$ and let $\tau_n \rightarrow \tau$. Consider first the case $\tau \in \mathbb{R}$. Since $1 + \tau Q(z)$ has no zero on $\overline{\mathcal{D}}$, it follows that

$$\inf_{z \in \mathcal{D}} |1 + \tau Q(z)| = d > 0.$$

The inverse triangle inequality

$$|1 + \tau_n Q(z)| \geq |1 + \tau Q(z)| - |\tau - \tau_n| |Q(z)|$$

together with the fact that $Q(z)$ is bounded on $\overline{\mathcal{D}}$ implies that for some n_0 the relation

$$\inf_{z \in \mathcal{D}} |1 + \tau_n Q_\tau(z)| \geq d/2$$

for all $n > n_0$ holds. Consequently

$$|Q_{\tau_n}(z) - Q_\tau(z)| = \frac{|\tau - \tau_n|}{|1 + \tau Q(z)| |1 + \tau_n Q(z)|} \leq 4d^{-2} |\tau - \tau_n|,$$

which means that $Q_{\tau_n}(z)$ converges locally uniformly to $Q_\tau(z)$ on \mathcal{D} . By [11, Theorem 1.4], $Q_{\tau_n}(z)$ converges locally uniformly to $Q_\tau(z)$ on $\mathbb{C}^+ \setminus \{\beta(\tau)\}$.

Now consider the case $\tau = \infty$. Since $\beta(\infty) = \alpha$, the function $Q(z)$ is bounded and bounded away from zero on $\bar{\mathcal{D}}$. Consequently, the locally uniform convergence $Q_{\tau_n}(z) \rightarrow Q_\infty(z)$ follows from

$$Q_{\tau_n}(z) - Q_\infty(z) = \frac{1}{1 + \tau_n Q(z)} \left(Q(z) + \frac{1}{Q(z)} \right) \rightarrow 0, \quad n \rightarrow \infty,$$

uniformly on \mathcal{D} and again [11, Theorem 1.4]. \square

Theorem 3.2. *The function $\tau \rightarrow \alpha(\tau)$ is continuous and the set*

$$\{\alpha(\tau): \tau \in \mathbb{R} \cup \{\infty\}\}$$

on the Riemann sphere is homeomorphic to a circle.

Proof. Let (τ_n) be some sequence which converges to τ . By Proposition 3.1, the sequence $(Q_{\tau_n}(z))$ converges locally uniformly to $Q_\tau(z)$ on $\mathbb{C}^+ \setminus \{\beta(\tau)\}$. Now it follows from [11, Theorem 1.4] that $\alpha(\tau_n) \rightarrow \alpha(\tau)$ if $n \rightarrow \infty$. This shows that the function $\tau \rightarrow \alpha(\tau)$ is continuous. Since it is also injective [15, Corollary 3.5] and the extended real line is compact on the Riemann sphere, the inverse of $\tau \rightarrow \alpha(\tau)$ is continuous as well. \square

Now the original topic about nonsymmetric extensions of Nevanlinna functions is taken up again.

Proposition 3.3. *Let $Q(z)$ be an \mathbf{N}_1 function. Assume that Ω is a simply connected domain with $\Omega \cap \mathbb{R} \neq \emptyset$ such that $\beta, \bar{\beta} \notin \Omega$, and assume that $Q(z)$ extends to a holomorphic function $\tilde{Q}(z)$ in $\Omega \cup \mathbb{C}^+$. If the set*

$$A = \{\alpha(\tau): \tau \in \mathbb{R} \cup \{\infty\}\} \cap \mathbb{R} \cap \Omega$$

has an accumulation point in Ω , then $\Omega \cap \mathbb{R}$ is outside the support of σ .

Proof. Consider the function

$$W(z) = \frac{\tilde{Q}(z) + \overline{\tilde{Q}(\bar{z})}}{2},$$

which is holomorphic in $\Omega \cup \mathbb{C}^+$ and real on $\Omega \cap \mathbb{R}$. Furthermore,

$$\tilde{Q}(z) = W(z), \quad z \in A,$$

since $Q(z) \in \mathbb{R}$ for $z \in A$. Hence, $Q(z) = W(z)$ for $z \in \Omega \cap \mathbb{C}^+$. In particular,

$$\lim_{z \rightarrow x} \operatorname{Im} Q(z) = 0, \quad x \in \mathbb{R} \cap \Omega$$

and therefore, $\Omega \cap \mathbb{R}$ is contained in the gap of $Q(z)$. \square

4. The behavior of $\alpha(\tau)$ meeting the real line

Proposition 4.1 below is a generalization of [15, Proposition 2.4] for the case when $z_0 = \alpha \in \mathbb{R}$ and the function $Q(z)$ extends holomorphically to a holomorphic function $\tilde{Q}(z)$ in some simply connected neighborhood Ω of α . Compared to [15], now it is not assumed that the extension satisfies the symmetry principle,

that is the α is not necessarily in the gap of the measure σ . The reasoning below is independent of [15], but in case $Q(z)$ satisfy the symmetry principle it reduces to the one in [15]. In what follows it will be frequently used that the k -th nontangential derivative of $Q(z)$ at α coincides with the derivative of the extension $\tilde{Q}(z)$ at α .

Proposition 4.1. *Let $Q(z) \in \mathbf{N}_1$ with $\alpha \in \mathbb{R}$ being its GZNT. If $Q(z)$ extends to a holomorphic function $\tilde{Q}(z)$ in some neighborhood Ω of α then precisely one of the following cases occurs:*

- (1) $\tilde{Q}'(\alpha) < 0$;
- (2) $\tilde{Q}'(\alpha) = 0$ and $\tilde{Q}''(\alpha) \neq 0$, in which case $\operatorname{Im} \tilde{Q}''(\alpha) \geq 0$;
- (3) $\tilde{Q}'(\alpha) = 0$ and $\tilde{Q}''(\alpha) = 0$, in which case $\tilde{Q}'''(\alpha) > 0$.

Proof. According to Theorem 2.2(ii) the extension $\tilde{Q}(z)$ can be represented as follows:

$$\tilde{Q}(z) = (z - \alpha)^2 R_0(z) \left(\frac{m_0}{\alpha - z} + \tilde{M}_1(z) \right), \quad (7)$$

with $m_0 \geq 0$ and

$$R_0(z) = 1/(z - \beta)(z - \bar{\beta}) \quad \text{or} \quad R_0(z) \equiv 1,$$

depending on the position of the GPNT. The function $\tilde{M}_1(z)$ in (7) is a Nevanlinna function in \mathbb{C}^+ which is also holomorphic in a neighborhood Ω of α of the form

$$\Omega = [\alpha - \varepsilon, \alpha + \varepsilon] + i[-\varepsilon, \varepsilon],$$

where $\varepsilon > 0$ is sufficiently small.

In order to list the possible cases, observe that it follows from (7) that

$$\tilde{Q}'(\alpha) = -m_0 R_0(\alpha), \quad R_0(\alpha) > 0.$$

Hence, $Q'(\alpha) \leq 0$.

If $Q'(\alpha) < 0$ case (1) prevails. Assume now that $\tilde{Q}'(\alpha) = 0$, in which case it follows from (7) that

$$\tilde{Q}''(\alpha) = 2R_0(\alpha)\tilde{M}_1(\alpha). \quad (8)$$

Since $\tilde{M}_1(z)$ is a Nevanlinna function in \mathbb{C}^+ and it is continuous at α it follows that $\operatorname{Im} \tilde{M}_1(\alpha) \geq 0$. Furthermore, note that $R_0(\alpha) > 0$. Therefore one has that

$$\tilde{Q}'(\alpha) = 0 \quad \Rightarrow \quad \operatorname{Im} \tilde{Q}''(\alpha) \geq 0, \quad (9)$$

which takes care of (2).

Finally, consider the case $\tilde{Q}'(\alpha) = \tilde{Q}''(\alpha) = 0$. Then by (8) and the fact that $R_0(\alpha) > 0$ one has $\tilde{M}_1(\alpha) = 0$. Consequently,

$$\tilde{Q}'''(\alpha) = 2R_0(\alpha)\tilde{M}_1'(\alpha). \quad (10)$$

Recall that by Theorem 2.1 the function $\tilde{M}_1(z)$ can be represented as

$$\tilde{M}_1(z) = \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \frac{\phi(t) dt}{t-z} + M_2(z),$$

where $\phi(z)$ is a function holomorphic in Ω , and $M_2(z)$ is a Nevanlinna function with a gap $[\alpha - \varepsilon, \alpha + \varepsilon]$. By the general theory of Nevanlinna functions [7] one has

$$\phi(\alpha) = \frac{1}{\pi} \operatorname{Im} \widetilde{M}_1(\alpha) = 0.$$

Furthermore, $\phi'(\alpha) = 0$, since $\phi(t)$ is positive on $[\alpha - \varepsilon, \alpha + \varepsilon]$. Hence, the function $\phi(t)/(\alpha - t)^2$ is integrable on $[\alpha - \varepsilon, \alpha + \varepsilon]$. By the dominated convergence theorem it follows that

$$\widetilde{M}_1'(\alpha) = \int_{\alpha - \varepsilon}^{\alpha + \varepsilon} \frac{\phi(t) dt}{(t - \alpha)^2} + M_2'(\alpha). \quad (11)$$

It is clear that $M_2'(\alpha) \geq 0$, since $M_2(z)$ has a gap at $[\alpha - \varepsilon, \alpha + \varepsilon]$. Hence (11) shows that $\widetilde{M}_1'(\alpha) \geq 0$. In fact, at least one of the terms in the right-hand side of (11) has to be positive, otherwise $Q(z) \equiv 0$, which is not an \mathbf{N}_1 function. Hence it follows that $M_2'(\alpha) > 0$ and, by (10), $\widetilde{Q}'''(\alpha) > 0$. This takes care of (3). \square

In [15, Theorem 4.1] it is investigated what cases can occur if the curve $\{\alpha(\tau) : \tau \in \mathbb{R} \cup \{\infty\}\}$ meets the real line in a spectral gap of the function $M(z)$. Either it approaches the spectral gap perpendicular and then continues through some subinterval of the spectral gap, or it approaches it with an angle of $\pi/3$, hits the spectral gap in a single point and leaves it with an angle of $2\pi/3$. However, if the curve meets the real line in a nonisolated point of $\operatorname{supp} \sigma$ its behavior might be dramatically different, as Example 5.3 in [15] and Section 5 below show. The following theorem provides a characterization of the possible cases in the situation that $M(z)$ has a (not necessarily symmetric) holomorphic extension through the corresponding point of intersection as in Proposition 4.1. The proof is an extension of the proof of [15, Theorem 4.1].

Theorem 4.2. *Let $Q(z) \in \mathbf{N}_1$ and assume that $Q(z)$ extends to a holomorphic function $\widetilde{Q}(z)$ in some neighborhood Ω of $z_0 \in \mathbb{R}$. Furthermore assume that $\alpha(\tau_0) = z_0$ for $\tau_0 \in \mathbb{R}$. Then precisely one of the following cases occurs:*

- (1) $Q'(z_0) < 0$. Then there exists $\varepsilon > 0$ such that the function $\alpha(\tau)$ is holomorphic on $(-\varepsilon, \varepsilon)$ and

$$\lim_{\tau \uparrow 0} \arg(\alpha(\tau) - z_0) = 0, \quad \lim_{\tau \downarrow 0} \arg(\alpha(\tau) - z_0) = \pi.$$

- (2) $Q'(z_0) = 0$ and $Q''(z_0) \neq 0$. Then there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that the function $\alpha(\tau)$ is holomorphic on each of the intervals $(-\varepsilon_1, 0)$ and $(0, \varepsilon_2)$. Moreover,

$$\lim_{\tau \downarrow 0} \arg(\alpha(\tau) - z_0) = \frac{2\pi - \theta_0}{2}, \quad \lim_{\tau \uparrow 0} \arg(\alpha(\tau) - z_0) = \frac{\pi - \theta_0}{2}, \quad (12)$$

where $\theta_0 = \arg Q''(z_0)$.

- (3) $Q'(z_0) = Q''(z_0) = 0$ and $Q'''(z_0) \neq 0$. Then there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that the function $\alpha(\tau)$ is holomorphic on each of the intervals $(-\varepsilon_1, 0)$ and $(0, \varepsilon_2)$. Moreover,

$$\lim_{\tau \uparrow 0} \arg(\alpha(\tau) - z_0) = \frac{\pi}{3}, \quad \lim_{\tau \downarrow 0} \arg(\alpha(\tau) - z_0) = \frac{2\pi}{3}.$$

Proof. Note that it is enough to consider the case $\tau = 0$. Indeed, if $Q(z)$ extends to some simply connected neighborhood Ω of z_0 then $Q_{\tau_0}(z)$ extends to $\Omega \setminus \{\beta(\tau_0), \overline{\beta(\tau_0)}\}$. Since $z_0 = \alpha(\tau_0) \notin \{\beta(\tau_0), \overline{\beta(\tau_0)}\}$ one can choose a sufficiently small Ω for the function $Q_{\tau_0}(z)$. In this situation the cases (1)–(3) correspond precisely to the classification in Proposition 4.1. Furthermore, without losing generality, it is assumed that $z_0 = 0$.

Case (1). According to the standard inverse function theorem, there exists a function $\phi(w)$ satisfying $Q(\phi(w)) = w$, cf. [15, Proof of Theorem 4.1]. Then define $\alpha(\tau) = \phi(\tau)$ for τ sufficiently small. The power series of $\tilde{Q}(z)$ at zero does need to have all its coefficients real, as was the case in [15, Proof of Theorem 4.1].

Case (2). $\tilde{Q}(0) = \tilde{Q}'(0) = 0$, and $\text{Im } \tilde{Q}''(0) > 0$. According to the generalized inverse function theorem, see e.g. [9, Theorem 9.4.3], the equation

$$\tilde{Q}(\phi^\pm(w)) = w^2,$$

has in some neighborhood of zero exactly two holomorphic solutions $\phi^+(w)$ and $\phi^-(w)$. The corresponding expansions

$$\phi^\pm(w) = \phi_1^\pm w + \phi_2^\pm w^2 + \dots,$$

satisfy $\phi_1^\pm = \pm(\tilde{Q}''(0)/2)^{-1/2}$, where the square root is chosen in such way that it transforms \mathbb{C}^+ onto itself. Recall that $\text{Im } \tilde{Q}''(0) \geq 0$ and, hence, it follows that $\pm \text{Im } \phi_1^\pm \geq 0$.

In the case $\tau > 0$ one has the identity

$$\tilde{Q}(\phi^-(\tau^{1/2})) = \tau \quad (13)$$

and $\arg \phi^- = \pi - \theta_0/2$. Hence $\phi^-(\tau^{1/2})$ is in $\mathbb{C}^+ \cup \mathbb{R}$ for small $\tau > 0$. As a consequence one sees that

$$\alpha(\tau) = \phi^-(\tau^{1/2}), \quad 0 < \tau < +\infty.$$

The expansion of $\phi^-(\tau^{1/2})$ implies the limit of $\arg(\alpha(\tau))$ as $\tau \downarrow 0$:

$$\lim_{\tau \downarrow 0} \tan(\arg(\alpha(\tau))) = \lim_{\tau \downarrow 0} \frac{\text{Im } \alpha(\tau)}{\text{Re } \alpha(\tau)} = \frac{\text{Im } \phi_1^-}{\text{Re } \phi_1^-} = \tan \arg \phi_1^- = \tan((2\pi - \theta_0)/2).$$

Since the tangent function is injective on the interval $[0, \pi]$, the first part of (12) follows.

Similarly, in the case $\tau < 0$ one has the identity

$$\tilde{Q}(\phi^+(i|\tau|^{1/2})) = -|\tau| = \tau, \quad (14)$$

and $\arg(\phi_1^+ i) = (\pi - \theta_0)/2$. Hence, $\phi^+(i|\tau|^{1/2})$ is in \mathbb{C}^+ for small $\tau < 0$. As a consequence one sees that

$$\alpha(\tau) = \phi^+(i|\tau|^{1/2}), \quad -\infty < \tau < 0.$$

The expansion of $\phi^+(i|\tau|^{1/2})$ implies the left limit of $\arg(\alpha(\tau))$ at zero:

$$\lim_{\tau \uparrow 0} \tan(\arg(\alpha(\tau))) = \lim_{\tau \uparrow 0} \frac{\text{Im } \alpha(\tau)}{\text{Re } \alpha(\tau)} = \tan(\arg(i\phi_1^+)) = \tan((\pi - \theta_0)/2).$$

Case (3) follows exactly along the same lines as in [15]. \square

5. Classification of GZNT

Let $Q(z)$ belong to \mathbf{N}_1 and assume for simplicity that its GPNT lies at infinity. Then the integral representation of $Q(z)$ has the following form:

$$Q(z) = (z - \alpha)(z - \bar{\alpha}) \left(a + bz + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\sigma(t) \right), \quad (15)$$

with $\alpha \in \mathbb{C}$, $a \in \mathbb{R}$, $b \geq 0$, and a measure σ satisfying $\int_{\mathbb{R}} d\sigma(t)/(t^2 + 1) < \infty$. If the GZNT α belongs to \mathbb{R} , then there is the following classification of α in terms of the integral representation (15):

- (A) $\delta_\alpha := \int_{\{\alpha\}} 1 d\sigma > 0$;
- (B) $\delta_\alpha = 0$, $\int_{\mathbb{R}} \frac{d\sigma(t)}{(t-\alpha)^2} = \infty$;
- (C) $\delta_\alpha = 0$, $\gamma_\alpha := \lim_{z \nearrow \alpha} \frac{Q(z)}{(z-\alpha)^2} \in \mathbb{R} \setminus \{0\}$, $\int_{\mathbb{R}} \frac{d\sigma(t)}{(t-\alpha)^2} < \infty$;
- (D) $\delta_\alpha = \gamma_\alpha = 0$, $\int_{\mathbb{R}} \frac{d\sigma(t)}{(t-\alpha)^2} < \infty$, $\int_{\mathbb{R}} \frac{d\sigma(t)}{(t-\alpha)^4} = \infty$;
- (E) $\delta_\alpha = \gamma_\alpha = 0$, $\int_{\mathbb{R}} \frac{d\sigma(t)}{(t-\alpha)^4} < \infty$,

cf. [5]. This classification has an interpretation in terms of a corresponding operator model, see [5, Theorem 5.1]. Furthermore, note that Proposition 4.1 provides an alternative classification of GZNT if $Q(z)$ has a (not necessarily symmetric) extension through α . Clearly, that classification is coarser than (A)–(E). Below the possible pairings are listed and examples of functions with symmetric and nonsymmetric extensions are indicated:

- (A) and (1): an example with a nonsymmetric extension is given by

$$Q(z) = z^2 \left(i - \frac{1}{z} \right),$$

see Example 5.1, while an example in the case of a spectral gap is given by $Q(z) = -z$.

- (B) and (2): an example with a nonsymmetric extension is given by

$$Q(z) = z^2 e^{i\theta_0}, \quad \theta_0 \in [0, \pi],$$

see Example 5.2, while examples in the case of a spectral gap do not exist.

- (C) and (2): an example with a nonsymmetric extension is given by

$$Q(z) = z^2 \left(1 + \int_{-1}^1 \frac{t^2 dt}{t-z} \right),$$

see Example 5.3, while an example in the case of a spectral gap is given by $Q(z) = z^2$.

- (D) and (3): an example with a nonsymmetric extension is given by

$$Q(z) = z^2 \left(\int_{-1}^1 \frac{t^2 dt}{t-z} \right),$$

see Example 5.4, while examples in the case of a spectral gap do not exist.

- (E) and (3): an example with a nonsymmetric extension is given by

$$Q(z) = z^2 \left(\int_{-1}^1 \frac{t^4 dt}{t-z} \right),$$

see Example 5.5, while an example in the case of a spectral gap is given by $Q(z) = z^3$.

The rest of this section is devoted to the treatment of these and other examples.

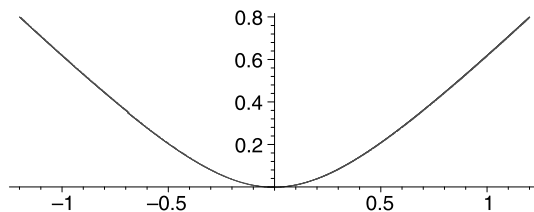


Fig. 1. Case (A), $Q(z) = z^2(i - 1/z)$.

Example 5.1. To illustrate Case (A) consider the \mathbf{N}_1 function

$$Q(z) = z^2 \left(i - \frac{1}{z} \right).$$

The plot in Fig. 1 (all plots obtained with Maple [12]) shows all the points z from the upper half-plane where $\text{Im } Q(z) = 0$. From Theorem 4.2 case (1) one can determine, that $\alpha(\tau)$ moves along the plotted curve from the right to the left hand side, i.e. $\text{Im Re}(\alpha(\tau))$ is decreasing in τ . Note that although $\alpha(\tau)$ approaches the origin horizontally, $\alpha(\tau) \notin \mathbb{R}$ for $\tau \neq 0$, in contrast to the case of a spectral gap described in [15]. This behavior agrees with [15, Theorem 3.6]. An example, which is simpler to compute, however not in the form (15), is

$$Q(z) = \frac{z^2}{(z-i)(z+i)} \left(i - \frac{1}{z} \right) = \frac{iz}{z-i}.$$

Solving

$$\frac{iz}{z-i} = \tau$$

one gets

$$\alpha(\tau) = \frac{-\tau + \tau^2 i}{\tau^2 + 1},$$

and the same effect of approaching the origin tangentially from both sides is obtained.

Example 5.2. To illustrate Case (B) consider for $\theta_0 \in [0, \pi]$ the function

$$Q(z) = z^2 e^{i\theta_0}.$$

Solving $Q(z) = \tau$ with $z \in \mathbb{C}^+$ one obtains

$$\alpha(\tau) = \begin{cases} \sqrt{|\tau|} \cdot e^{i(\pi-\theta_0)/2}, & \tau \leq 0, \\ \sqrt{\tau} \cdot e^{i(2\pi-\theta_0)/2}, & \tau > 0. \end{cases}$$

As another example of Case (B) consider

$$Q(z) = z^2 \int_{-1}^1 \frac{dt}{t-z}.$$

Fig. 2 contains the plot of points $z \in \mathbb{C}^+$ satisfying $\text{Im } Q(z) = 0$. Since $\alpha(\infty) = \infty$ one sees that for sufficiently small $\tau < 0$ the point $\alpha(\tau)$ moves with increasing τ along the real line to the left until it reaches

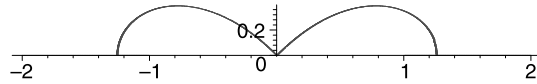


Fig. 2. Case (B), $Q(z) = z^2 \int_{-1}^1 \frac{dt}{t-z}$.

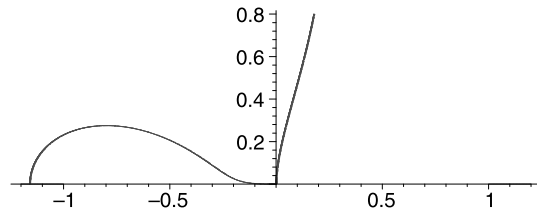


Fig. 3. Case (C), $Q(z) = z^2 (1 + \int_{-1}^1 \frac{t^2 dt}{t-z})$.

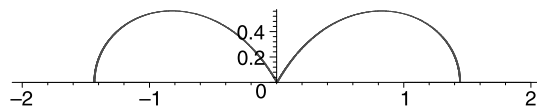


Fig. 4. Case (D), $Q(z) = z^2 (\int_{-1}^1 \frac{t^2 dt}{t-z})$.

the point near 1.7. There it leaves the real line to the upper half-plane and continues along the plotted path until it reaches the real line again at approximately -1.7 . From that point it continues along the real line.

Example 5.3. To illustrate Case (C) consider the function

$$Q(z) = z^2 \left(1 + \int_{-1}^1 \frac{t^2 dt}{t-z} \right).$$

Fig. 3 contains the plot of points $z \in \mathbb{C}^+$ satisfying $\text{Im } Q(z) = 0$. One may observe that with $\tau \downarrow 0$ the point $\alpha(\tau)$ approaches the real line approximately vertically and it leaves the origin approximately horizontally. It is known from Theorem 3.6 of [15] that the only point in a neighborhood of zero where $\alpha(\tau) \in \mathbb{R}$ is the origin itself.

Example 5.4. To illustrate Case (D) consider the function

$$Q(z) = z^2 \int_{-1}^1 \frac{t^2 dt}{t-z}.$$

Fig. 4 contains the plot of points $z \in \mathbb{C}^+$ satisfying $\text{Im } Q(z) = 0$. Note the essential difference between Fig. 2 and Fig. 4, in Fig. 2 the angle between the left and right limit of $\alpha(\tau)$ at the origin is $\pi/2$, while in Fig. 4 it is $\pi/3$. The movement of $\alpha(\tau)$ along the plotted line and the real line is the same as in Example 5.2.

Example 5.5. Finally, to illustrate Case (E) consider the function

$$Q(z) = z^2 \left(\int_{-1}^1 \frac{t^4 dt}{t-z} \right).$$

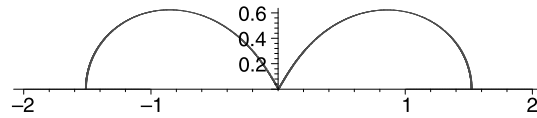


Fig. 5. Case (E), $Q(z) = z^2 \left(\int_{-1}^1 \frac{t^4 dt}{t-z} \right)$.

Fig. 5 contains the plot of points $z \in \mathbb{C}^+$ satisfying $\text{Im } Q(z) = 0$. The angle between the left and right limit of $\alpha(\tau)$ at the origin is $\pi/3$ and the movement of $\alpha(\tau)$ along the plotted line and the real line is the same as in Examples 5.2 and 5.3.

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